Logical Completeness of Differential Equations

Long Qian
Carnegie Mellon University, USA

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Differential equations are fundamental across many disciplines, frequently used in modelling systems with continuous dynamics. It is therefore important to be able to correctly prove properties of differential equations, especially in safety-critical situations. Concretely, let $x' = f(x)$ be a n-dimensional differential equation with $\phi(x, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ denoting its flow (assuming global existence for brevity). For a set of initial values $I \subseteq \mathbb{R}^n$ and a time interval $[0, T]$, common properties of interest include:

- Safety: Let $S \subseteq \mathbb{R}^n$ be a set of safe values, is it true that for every $x_0 \in I$, the trajectory of x_0 following $x' = f(x)$ stays in S on the time interval $[0, T]$? I.e. is the following formula valid?

$$\forall x_0 \in I \, \forall t \in [0, T] \; \varphi(x_0, t) \in S$$

- Liveness: Let $G \subseteq \mathbb{R}^n$ be a set of goal values, is it true that for every $x_0 \in I$, the trajectory of x_0 following $x' = f(x)$ reaches G on the time interval $[0, T]$? I.e. is the following formula valid?

$$\forall x_0 \in I \, \exists t \in [0, T] \; \varphi(x_0, t) \in G$$

One approach to validating such properties is to proceed quantitatively, computing numerical approximations to the reachable sets. However, the correctness of such methods are generally difficult to guarantee due to their numerical nature. Alternatively, one can also take a more qualitative approach, where a (small) set of general axioms concerning differential equations are proven to be valid, and properties of ODEs are established deductively by iteratively applying such axioms. Consequently, the correctness of such proofs only depend on the validity of a small set of core axioms, and can be independently verified by theorem provers implementing this logical framework [1]. This logical system is called *differential dynamic logic* [2].

However, as such qualitative axioms are symbolic and qualitative, they are seemingly less capable than quantitative approaches at validating inherently numerical properties of differential equations. Naturally, one can ask for which class of properties can such qualitative axioms validate? Equivalently, is differential dynamic logic *complete* for certain class of properties? In joint work with André Platzer [3], we show that completeness for safety and liveness properties hold when the sets in question I, S, G are all first-order definable, I is compact and S, G both open.

References

[1] Nathan Fulton, Stefan Mitsch, Jan-David Quesel, Marcus Völp, and André Platzer. 2015. KeYmaera X: An Axiomatic Tactical Theorem Prover for Hybrid Systems. In *Automated Deduction - CADE-25 - 25th International Conference on Automated Deduction, Berlin, Germany, August 1-7, 2015, Proceedings (LNCS, Vol. 9195)*, Amy P. Felty and Aart Middeldorp (Eds.). Springer, Berlin, Germany, 527–538. https://doi.org/10.1007/978-3-319-21401-6_36

[2] André Platzer. 2008. Differential Dynamic Logic for Hybrid Systems. *J. Autom. Reason.* 41, 2 (2008), 143–189. https://doi.org/10.1007/S10817-008-9103-8

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