Families of eulerian functions involved in regularization of divergent polyzetas

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For any $r \in \mathbb{N}_{\geq 1}$ and $(s_1, \ldots, s_r) \in \mathbb{C}^r$, let us consider the following *several variable zeta function* (polyzetas) [3] $\zeta(s_1, \ldots, s_r) := \sum_{n_1 > \ldots > n_r > 0} n_1^{-s_1} \ldots n_r^{-s_r}$ which converges for $(s_1, \ldots, s_r)$ in the open sub-domain of $\mathbb{C}^r$ [2] and [6], $\mathcal{H}_r := \{(s_1, \ldots, s_r) \in \mathbb{C}^r \,|\, \forall m = 1, \ldots, r, \sum_{i=1}^{m} \mathrm{Re}(s_i) > m\}$. From Weierstrass factorization and Newton-Girard identity [3] and [4], one has successively

$$\frac{1}{\Gamma(z+1)} = e^{\gamma z} \prod_{n \geq 1} \Big(1 + \frac{z}{n}\Big) e^{-\frac{z}{n}} = \exp\Big(\gamma z - \sum_{k \geq 2} \zeta(k) \frac{(-z)^k}{k}\Big) \tag{1}$$

where $\Gamma(z)$ defines the Gamma function. One can deduce the following expression for $\zeta(2k)$:

$$\frac{\zeta(2k)}{\pi^{2k}} = k \sum_{l=1}^{k} \frac{(-1)^{k+l}}{l} \sum_{\substack{n_1, \ldots, n_l \geq 1 \\ n_1 + \ldots + n_l = k}} \prod_{i=1}^{l} \frac{1}{\Gamma(2n_i + 2)} \in \mathbb{Q}. \tag{2}$$

The formula (2) is a different version of a result of L. Euler using Bernoulli numbers

$$\frac{\zeta(2k)}{\pi^{2k}} = \frac{(-1)^{k+1} 2^{2k-1} B_{2k}}{(2k)!}, \; k \in \mathbb{N}.$$

In this talk, based on the combinatorics of noncommutative generating series, we discuss a way to extend the formula (1) and then we present a recurrence relation of $\zeta(2^k, \ldots, 2^k), \; k \in \mathbb{N}^*$. This is based on join works with Prof. Hoang Ngoc Minh and Prof. Gérard Duchamp [4].

References

[1] Jean Dieudonné, *Infinitesimal calculus*, Houghton Mifflin, 1971.
[2] A.B. Goncharov, *Multiple polylogarithms and mixed Tate motives*, 2001.
[3] V. Hoang Ngoc Minh, *Summations of Polylogarithms via Evaluation Transform*, in Math. & Comp. in Simul., 1336, p. 707-728, 1996.
[4] Bui Van Chien, Hoang Ngoc Minh, Ngo Quoc Hoan, Nguyen Dinh Vu, *Families of eulerian functions involved in regularization of divergent polyzetas*, Pub. Math. de Besancon, p. 5-28, 2023.
[5] A. Lascoux, *Fonctions symétriques*, SLC, B08e, 1983.
[6] J. Zhao, *Analytic continuation of multiple zeta functions*, Proc. A. M. S., 128 (5), p. 1275 - 1283, 1999.