Goncharov's programme, and symmetries of weight 6 multiple polylogarithms

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Multiple polylogarithms $\mathrm{Li}_{k_1,\ldots,k_d}(x_1,\ldots,x_d)$ are a class of multi-variable special functions generalising the natural logarithm $\mathrm{Li}_1(x) = -\log(1-x)$. These functions appear in connection with K-theory, hyperbolic geometry, values of L-functions, mixed Tate motives, high-energy physic, and many other areas.

One of the main challenges in the study of MPL's revolves around understanding on how many variables a MPL (or 'interesting' combinations thereof) actually depend ("the depth"). It is well known, for example, that $\mathrm{Li}_{1,1}$ can already be expressed via Li_2, likewise $\mathrm{Li}_{1,1,1}$ can be expressed via Li_3. Goncharov gave a conjectural criterion ("the Depth Conjecture") to determine this, using the motivic coproduct, as part of his programme to investigate Zagier's Polylogarithm Conjecture on the special values of the Dedekind zeta function $\zeta_F(m)$.

I will give an overview of Goncharov's Depth Conjecture, and its implications. I will discuss what is currently known, including recent progress in weight 6. In particular, the conjecture predicts that a certain weight 6 function (essentially a small modification of $\mathrm{Li}_{4,1,1}(x,y,z)$) should satisfy the 6-fold dilogarithm symmetries $\lambda \mapsto \lambda^{-1}, 1-\lambda$ in each variable independently, modulo depth ≤ 2 terms.

I will then describe the computational background and tools involved in investigating and proving these symmetries. In particular, one has to consider many possible degenerations (to boundary components of $\mathfrak{M}_{0,n}$) of the Matveiakin-Rudenko quadrangular polylogarithm functional equations, to iteratively find weaker symmetries of $\mathrm{Li}_{4,1,1}$ and useful short identities. To investigate higher weight analogues will require a more structure approach and understanding of this degeneration process.