New Results about Bricard’s Flexible Octahedra

Robert H. Lewis[1], Mosavverul Hassan[2] and Evangelos Coutsias[3]
[1] Fordham University, NY, USA
[2] Vanderbilt University, TN, USA
[3] Stony Brook University, NY, USA

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Biological functions such as signal transduction, enzymatic turnover, and allosteric regulation emerge as a consequence of protein conformational transitions (protein dynamics) across a complex energy landscape. Illuminating the mechanistic basis of protein function requires an understanding of why structures such as molecules can become flexible.

A polypeptide backbone can be modeled as a polygonal line whose edges and angles are fixed while some of the dihedral angles formed by successive triplets of edges vary freely, so that the structure is flexible. We model and analyze such a structure with a system of polynomial equations.

This subject has a long history.

- In 1812, Cauchy considered flexibility of three dimensional polyhedra (edges and faces), where each joint can pivot or hinge. He proved [2] that if the polyhedron is convex it must be rigid – i.e. cannot be flexible.
- In 1896 Bricard [1], following Cauchy, found three types of flexible non-convex octahedra, but the faces intercross, at least in 3-space.
- Following Bricard’s ideas, Connelly (1978) found non-convex genuine flexible polyhedra [3] that really live in 3-space.

In spite of that success, key questions remained. Bricard asserted that a certain planar configuration of quadrilaterals can be flexible in the same way as the octahedra, since both systems satisfy the same set of polynomial equations:

$$A_1 t^2 u^2 + B_1 t^2 + C_1 tu + D_1 u^2 + E_1,$$
$$A_2 u^2 v^2 + B_2 u^2 + C_2 uv + D_2 v^2 + E_2,$$
$$A_3 v^2 t^2 + B_3 v^2 + C_3 vt + D_3 t^2 + E_3$$

Here the variables t, u, v represent angles and the coefficients are polynomials in the edges. The geometric structure is flexible if this system of polynomial equations has infinitely many solutions. In 2016 [4] we used computer algebra to show that the quadrilaterals have additional modes for flexibility. We did that by analyzing the Dixon resultant [5] of the system.

Another statement from Bricard [1], which has been called the Bricard conjecture, has remained unjustified until now:

Conjecture: The system of three equations above has infinitely many solutions iff t, u, and v satisfy both of these equations:

$a_1 t + b_1 u + c_1 v + d_1 tuv = 0,$
$a_2 tu + b_2 tv + c_2 uv + d_2 = 0,$
where the coefficients are polynomials in the edges. The "if" part here is easy, a simple exercise. The converse has never been proven.

Main Result: The converse is true for every known case of flexible structures. That includes the three types of flexible octahedra and every known case of the flexible planar quadrilaterals. The proof is with computer algebra, normalizing the 8×8 Dixon resultant, which contains polynomials in twelve variables with up to 100000 terms.

Secondary Result: As a biproduct of the main result, we have produced animations of Bricard's type three flexible octahedron. Apparently this has never been done before.

Keywords: flexible structures, octahedron, computer algebra

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