Toric ideals of graphs minimally generated by a Gröbner basis

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The problem of describing families of ideals minimally generated by either one or all of its Gröbner bases is a central topic in commutative algebra. This work tackles this problem in the context of toric ideals of graphs. We call a graph G an MG-graph if its toric ideal I_G is minimally generated by a Gröbner basis, while we say that G is an UMG-graph if every reduced Gröbner basis of I_G is a minimal generating set.

We prove that G is an UMG-graph if and only if I_G is a generalized robust ideal, i.e., ideal whose universal Gröbner basis and universal Markov basis coincide. We observe that the class of MG-graphs is not closed under taking subgraphs, and we prove that it is hereditary (i.e., closed under taking induced subgraphs). Also, we describe two families of bipartite MG-graphs: ring graphs and graphs whose induced cycles have the same length. The latter extends a result of Ohsugi and Hibi, which corresponds to graphs whose induced cycles have all length 4.

While working on this project, we have been making intensive and constant use of the software SageMath to generate examples and support conjectures. We also used the software SageMath for computing the whole Gröbner fan of the corresponding toric ideal, and thus we can only handle small examples in a reasonable amount of time. We have used the above computations together with the Nauty library to check that the only bipartite graph with ≤ 8 vertices that is not an MG-graph is the cube graph (the 1-skeleton of the 3-dimensional cube).

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