Symbolic integration on a planar differential foliation

Thierry Combot
University of Bourgogne, France

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In this presentation, we will study the differential algebraic properties of integrals of the form $\int G(x, y(x, h))\,dx$ where y is a family of solutions of differential equations.

Consider a differential equation $y' = F(x, y)$ with F rational. This equation defines a foliation of the plane $\mathcal{F}$, and we consider the integral $\int_{\mathcal{F}} G(x, y(x))\,dx$ along the leaves of $\mathcal{F}$, with G rational. Alternatively, we can write $I(x, h) = \int G(x, y(x, h))\,dx$ where $y(x, h)$ is a family of solutions of $y' = F(x, y)$. if $\mathcal{F}$ is an algebraic foliation, such integral is D-finite and is always differentially algebraic in h. Oppositely, let us assume that $y' = F(x, y)$ has no rational first integral. We will prove that if $I(x, h)$ is differentially algebraic, then, up to parametrization change in h, $I(x, h)$ satisfies a differential equation of the form $LI = (\partial y(x, h))^{\ell} H(x, y(x, h))$ where $L \in \mathbb{C}[\partial_h]$ has constant coefficients. The possible operators L depends on the existence of an integrating factor for $y' = F(x, y)$ and its algebraic nature. We will present an efficient algorithm to find such minimal integrating factor. We will then present an algorithm to find a differential relation up to some given bound on the order of L and degree of H. In the particular case of the foliations $y = \ln x + h$ et $\ln y = \alpha \ln x + h$, we have a complete algorithm to decide if integrals are differential algebraic, and this leads to explicit formulas in terms of special functions Ei, Li, Φ. This allows to study the differential transcendence of the flow of a differential equation in the plane. If possible, we will present how this generalizes in higher dimension.

References

[1] Thierry Combot. Symbolic integration on a planar differential foliation, 21 Jun 2023. https://arxiv.org/abs/2306.12573.