Conservative Matrix Fields - Algebra and Asymptotics

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D-finite sequences, also known as Holonomic or P-recursive sequences, are a family of special functions, which are ubiquitous in many areas of mathematics. The asymptotic properties of these sequences were detailed in landmark papers by Poincare [1] and Perron [2]. Notably, ratios of D-finite sequences satisfying the same recurrence, also known as Apéry limits, are at the core of many irrationality results [3] to [7]. However, finding such sequences with desirable limits and irrationality measures remains a challenge.

In this talk we introduce an interesting object, the Conservative Matrix Field. This object has been used in identity proofs [8], in Diophantine approximations, [9], and most recently for unifying hundreds of formulas for π [10] (see Figure 1). We will discuss how this object generates a high dimensional generalization of Apéry limits, by deriving such a sequence from each rational direction in $\mathbb{R}^d$. This generalization keeps the desirable properties of Apéry limits, yet simplifies the search for useful ones. More concretely, experimental analysis suggests the irrationality measure of the Apéry limits is continuous with respect to direction, while the actual sequence limit remains constant (see Figure 2). This surprising phenomenon allows for optimization-based search algorithms, such as gradient descent, to be used in the search for irrationality proving approximations.

We will present how Conservative Matrix Fields can be constructed using ideals of finite codimension in an Ore algebra (such as annihilators of D-finite functions), as well as their interesting phenomenological properties. Finally, we will update about our currently ongoing effort to prove these properties.

References

[1] Henri Poincaré. Sur les équations linéaires aux différentielles ordinaires et aux différences finies. *American Journal of Mathematics*, 1885.

[2] Oskar Perron. Über Summengleichungen und Poincarésche Differenzengleichungen. *Mathematische Annalen*, 1921.

[3] Roger Apéry. Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque*, 1979.

[4] Marc Chamberland and Armin Straub. Apéry Limits: Experiments and Proofs. *The American Mathematical Monthly*, 2021.

[5] Francis Brown and Wadim Zudilin. On cellular rational approximations to $\zeta(5)$. `arXiv:2210.03391`, 2022.

[6] Wadim Zudilin. An Apéry-like difference equation for Catalan's constant. `arXiv:math/0201024`, 2002.

[7] A. I. Aptekarev. On linear forms containing the Euler constant. `arXiv:0902.1768`, 2009.

[8] William Gosper. Strip Mining in the Abandoned Orefields of Nineteenth Century Mathematics *Computers In Mathematics*, 1990.

[9] Rotem Elimelech et al. Algorithm-assisted discovery of an intrinsic order among mathematical constants. *Proceedings of the National Academy of Sciences*, 2024.

[10] Tomer Raz et al. From Euler to AI: Unifying Formulas for Mathematical Constants. `arXiv:2502.17533`, 2025.

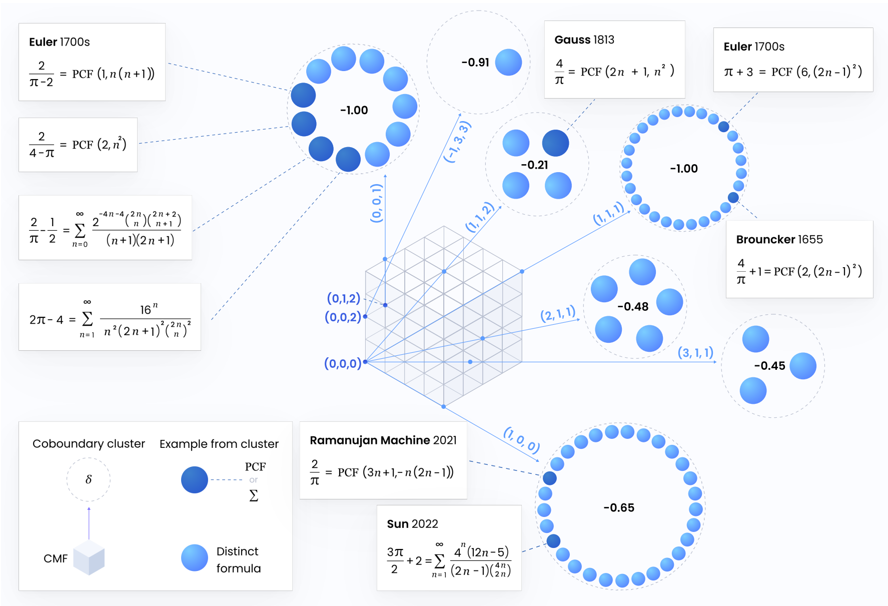

Figure 1: **Formula unification by a Conservative Matrix Field**. Numerous π formulas harvested from the literature are automatically arranged as directions in a Conservative Matrix Field defined over $\mathbb{Z}^3$. These formulas include famous ones by Gauss, Euler, and Lord Brouncker. More details available at [10]

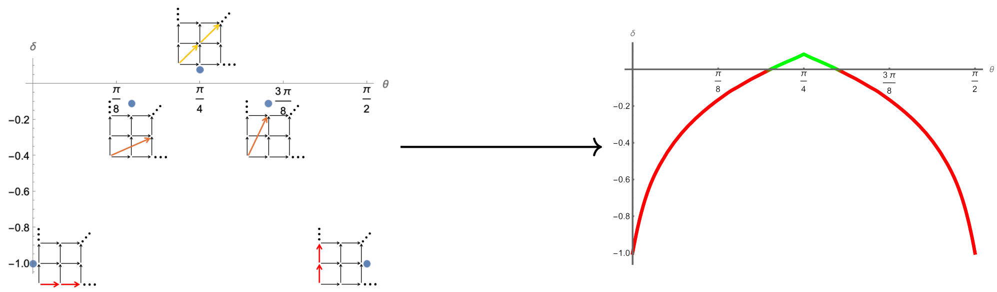

Figure 2: **Demonstration of the continuity phenomenon of the irrationality measure**, shown for a Conservative Matrix Field defined on $\mathbb{Z}^2$; the sequences resulting from it converge to $\zeta(3)$. **Left:** a graph of 5 angles, and the estimated irrationality measure of the Diophantine approximations associated with them in the Conservative Matrix Field. Over each data point is a sketch of the angle in $\mathbb{Z}^2$. **Right**: an interpolation of the irrationality measures of a few dozen different angles, demonstrating its surprising continuity. Positive values of δ are in green as they indicate these directions generate sequences that prove irrationality.