Non-commutative D-finite & D-algebraic power series and formal languages

Robert Green and Joshua Grochow
University of Colorado, USA

30th Applications of Computer Algebra - ACA 2025

We define and study non-commutative analogues of multivarite D-finite and D-algebraic generating functions, and the complexity classes of languages corresponding to them. In particular, we give both equational (fixed point) characterizations and automata machine model characterizations of these classes, and relate them to standard language classes (regular, linear, context-free, and tree-adjoining). We prove several inclusions and separations between our new classes and each other, and our new classes and classical language classes. Among our more surprising results are:

- Left D-finite and right D-finite languages are not the same, unlike the case of left linear and right linear (which both give exactly the regular languages).

- There are non-commutative algebraic power series (corresponding to CFLs) that are not D-finite, in contrast to the classical theorem that commutative algebraic power series are D-finite.

- There are left D-finite languages that are not even tree-adjoining, and there are tree-adjoining languages that are not even D-algebraic.

In addition to proving many results on these classes, we also highlight many open questions ripe for future research.